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Restricted Systems of Equations.[*]

By Arthur B. Coble.

Throughout the following account geometric ideas and language are employed exclusively, though the material is algebraic. A homogeneous equation of degree l in $n+1$ variables is called a spread of order l in S_n. A manifold of dimension r is indicated by M_r. According to Kronecker[†] it can always be defined as the totality of points on at most $n+1$ spreads. The simplest problem in the theory of restricted systems of equations is the following: Given n spreads in S_n of orders $\lambda_1, \ldots, \lambda_n$, all of which contain M_r, in how many points of S_n outside of M_r do they meet? A very wide range of problems in enumerative geometry and algebra can be expressed in terms of restricted systems of equations. With this in mind, Salmon[‡] began to develop a theory of such equations. His method was inductive, but the general case was not touched. Furthermore, the results which he obtained were proved only for the case where the manifold M_r is the complete intersection of $n-r$ spreads in S_n, a manifold of a type referred to hereafter as "regular." The examples given at the end of § 2 show that such results may or may not be true for the general case. That no systematic development of the subject of restricted systems has been attempted is due, no doubt, to the success of Schubert[§] and others in the field of enumerative geometry with the aid of the principle of the conservation of number and a certain symbolic calculus.

The object of this paper — the first of a series under the same title — is to give a general account of the theory of restricted systems of equations. The simple problem formulated above is considered in § 1. A solution is obtained in terms of the orders λ, and of $r+1$ so-called "index numbers of M_r in S_n" which depend only on the M_r in S_n. Certain theorems relating to the determination of these index numbers are derived, and applications to specific problems are made.

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[†] *Crelle*, 92.

[‡] "Lessons on Modern Higher Algebra." Cf. E. Lasker, "Zur Theorie der Moduln und Ideale," *Matematische Annalen*, Vol. LX (1905), in particular p. 44 and p. 112.

[§] "Kalkül der abzählenden Geometrie," Leipzig (1879); cf. Pascal-Schepp, *Repertorium*, Vol. II, Chap. XV (1902).

In § 2 there is treated the more general problem in which S_n is replaced by a manifold M_n in a linear space of any dimension greater than n. For this case the $r+1$ "relative index numbers of M_r as to M_n" are introduced. It is shown that they have properties entirely analogous to those of the ordinary index numbers. Some relations connecting the two kinds of index numbers are also given.

Further problems are readily suggested, and these will be discussed in later papers. Possibly the most important is that of the "incomplete restricted system," namely: Given $n-r+k$ spreads of orders $\lambda_1, \ldots, \lambda_{n-r+k}$ on M_r in S_n which meet in a residual M_{r-k} which has an M_{r-k-1} in common with M_r; what are the index numbers of M_{r-k} and of M_{r-k-1} in S_n, and what are the relative index numbers of M_{r-k-1} as to M_r and as to M_{r-k}? For $k=r$ we have again the problem of § 1. An obvious generalization is the problem of the "incomplete relative restricted system." It appears that if $r \geqq 2+k$, the ordinary index numbers of M_r are not sufficient for the solution and further index numbers must be introduced. Closely related to the above inquiry is that as to the index numbers of a composite manifold in terms of those of its constituent manifolds.

Beginning with the fact that k manifolds in S_n of orders $\lambda_1, \ldots, \lambda_k$ and dimensions $r_1, \ldots, r_k$, where $\sum\limits_i r_i = n(k-1)$, ordinarily meet in $\prod\limits_{i=1}^{k} \lambda_i$ points, we may ask what is the reduction in this number due to the fact that the manifolds all contain M_r, $r < r_i$. This question can be extended in the same directions as the one originally put, and doubtless leads to the most general one in the subject.

Some attention will be devoted to the determination of the index numbers of certain spreads, such as those defined by matrices and those which occur in mapping.

§ 1. *The Index Numbers of a Spread in a Linear Space.*

1. Let M_r be a manifold of dimension r in a space S_n whose section by an arbitrary $S_{n-(r-k)}$, $0 \leqq k \leqq r$, is M_k. In $S_{n-(r-k)}$ let M_k be on $n-(r-k)$ spreads of dimension $n-(r-k)-1$ and of orders $\lambda_1, \lambda_2, \ldots, \lambda_{n-(r-k)}$ which meet further in a finite number, O_k, of points not on M_k. If σ_j be the sum of the products of the λ's, j at a time, then *the* $(k+1)$*-th index number of* M_r *in* S_n is defined by the formula

$$(1) \quad \alpha_k = \sigma_{n-(r-k)} - \alpha_0\sigma_k - \alpha_1\sigma_{k-1} - \alpha_2\sigma_{k-2} - \ldots - \alpha_{k-1}\sigma_1 - O_k,$$

in terms of the orders λ, the number O_k, and the earlier index numbers α_0, α_1,

$\ldots, \alpha_{k-1}$, which are similarly defined by means of sections of M_r in S_{n-r}, $S_{n-r+1}, \ldots, S_{n-(r-k+1)}$. In particular, α_0 is the *order* of M_r. The *last* index number, α_r, is defined by n spreads of orders λ_i in S_n and this particular form of (1):

$$(2) \qquad \alpha_r = \sigma_n - \alpha_0\sigma_r - \alpha_1\sigma_{r-1} - \alpha_2\sigma_{r-2} - \ldots - \alpha_{r-1}\sigma_1 - O_r;$$

and it is convenient to say that

(3) *The $r+1$ index numbers of M_r in S_n are the last index numbers of M_r and its successive linear sections.*

For a given set of index numbers, $\alpha_0, \alpha_1, \ldots$, and a given set of orders, $\lambda_1, \ldots, \lambda_h$, we shall often use a symbol A_j defined by

$$(4) \qquad A_j = \alpha_0\sigma_j + \alpha_1\sigma_{j-1} + \alpha_2\sigma_{j-2} + \ldots + \alpha_{j-1}\sigma_1 + \alpha_j.$$

Then, if σ' and A' refer only to the orders $\lambda_2, \ldots, \lambda_h$, we have

$$(5) \qquad \sigma_j = \sigma'_j + \lambda_1\sigma'_{j-1}, \quad A_j = A'_j + \lambda_1 A'_{j-1}.$$

2. The index numbers defined above are independent of the orders of the spreads employed in the definition. This is certainly true of the first index number α_0, the order of M_r, and according to (3) needs to be proved only for the last index number α_r. Beginning with (2), which we write $O_r = \sigma_n - A_r$, let the order λ_1 of the spread f_1 be increased by unity by adding to f_1 an arbitrary linear spread L which cuts M_r in M_{r-1}. Then $O''_r = \sigma''_n - A''_r$, where the superscript refers to the new order λ_1+1 and the possible new index number α''_r. The points O''_r outside of M_r are made up of the points O_r and the points O'_{r-1} outside of M_{r-1} in L. But $O'_{r-1} = \sigma'_{n-1} - A'_{r-1}$, where the superscript refers to the orders $\lambda_2, \ldots, \lambda_n$ and the index numbers $\alpha_0, \ldots, \alpha_{r-1}$. Thus $O''_r = O_r + O'_{r-1} = \sigma_n - A_r + \sigma'_{n-1} - A'_{r-1} = \sigma''_n - A_r - A'_{r-1}$. Hence $A''_r = A_r + A'_{r-1}$, and we see from (5) that $\alpha''_r = \alpha_r$. By induction the index numbers are independent of any increase in any of the orders due to the addition of linear spreads. According to Schubert's principle of the conservation of number, they are independent of any possible increase. Since the original orders might have been the lowest possible ones, the index numbers are entirely independent of the orders. The application of Schubert's principle here is eminently proper. We may regard the spread M_r as defined by $f_1, \ldots, f_n$ and the exclusion of the O_r outside points. Then $f_1 L, \ldots, f_n$, with the exclusion of O''_r, define M_r in precisely the same way for our present purpose. To be sure, $f_1 L$ has M_{r-1} as a locus of double points at least while f_1 itself may have only simple points on M_{r-1}. But this does not affect the number of outside intersections. For in both cases $f_2, \ldots, f_n$ meet in M_r and a residual curve which cuts M_r in γ points.

To obtain the legitimate number of outside intersections, we have only to choose L so that it contains none of these γ points.

(6) *The index numbers of* M_r *in* S_n *do not depend upon the orders of the spreads used to define them. They depend only on* M_r *itself and the dimension in which* M_r *lies.*

The dependence upon the dimension is given below in (14).

3. From (6) and (2) we have at once that

(7) *The order of a restricted system of equations in* S_n *with the common solution* M_r *is given by the formula*

$$O_r = \sigma_n - \alpha_0 \sigma_r - \alpha_1 \sigma_{r-1} - \alpha_2 \sigma_{r-2} - \dots - \alpha_{r-1} \sigma_1 - \alpha_r = \sigma_n - A_r,$$

in which $\alpha_0, \dots, \alpha_r$ *are the index numbers of* M_r *in* S_n *and the* σ*'s refer to the orders of the given equations.*

Thus if the order of one restricted system for M_r itself and for each of its successive sections is known and thereby the α's are determined, then the order of *any* restricted system involving M or any of its linear sections can be obtained. The first problem in the theory of restricted systems is the determination of the index numbers of given spreads. Some theorems relating to this will now be derived.

Let us call M_r in S_n a *complete manifold of excess* k if it is the complete intersection of $n-r+k$ $(0 \leqq k \leqq r)$ spreads in S_n. If, in particular, $k=0$, M_r will be called a *regular manifold*; an *ordinary manifold*, if it lies in an S_{r+1} in S_n.

4. Let then M_r be a complete manifold of excess k defined in S_n by spreads of orders $\lambda_1, \lambda_2, \dots, \lambda_{n-r+k}$, of which λ_1 is the maximum. In (1), $O_k = 0$, and α_k is determined in terms of the given orders and the earlier index numbers by the formula $0 = \sigma_{n-r+k} - A_k$. In $S_{n-r+k+1}$ an additional spread is required to determine α_{k+1}. This can always be taken equal to λ_1 in such a way that $O_{k+1} = 0$, and for sections of greater dimension further spreads of order λ_1 can be used. By repeated application of (4) we can write the equations defining the index numbers beyond α_{k-1} as follows, the σ's referring of course to the $n-r+k$ given orders:

$$\begin{aligned}
0 &= \sigma_{n-r+k} - [A_k], \\
0 &= \lambda_1 \sigma_{n-r+k} - [\lambda_1 A_k + A_{k+1}], \\
0 &= \lambda_1^2 \sigma_{n-r+k} - [\lambda_1^2 A_k + 2\lambda_1 A_{k+1} + A_{k+2}], \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\
0 &= \lambda_1^j \sigma_{n-r+k} - \Big[\lambda_1^j A_k + \binom{j}{1} \lambda_1^{j-1} A_{k+1} + \binom{j}{2} \lambda_1^{j-2} A_{k+2} + \dots \\
&\qquad + \binom{j}{j-1} \lambda_1 A_{k+j-1} + \binom{j}{j} A_{k+j}\Big].
\end{aligned}$$

Multiplying these equations in order, beginning with the last, by the terms in the development of $(1-\lambda_1)^j$ and using the formula

$$(8)\quad \binom{a}{0}\binom{a+b}{c}-\binom{a}{1}\binom{a+b-1}{c}+\binom{a}{2}\binom{a+b-2}{c}-\dots\dots$$
$$+(-1)^{a+b-c}\binom{a}{a+b-c}\binom{c}{c}=\begin{cases}0 \text{ if } c<a,\\ 1 \text{ if } c=a,\end{cases}$$

we get $A_{k+j}=0$, if $j>0$. Hence the index numbers, beginning with α_k, are determined by the equations

$$(9)\qquad A_k=\sigma_{n-r+k},\quad A_{k+j}=0,\qquad j=1,2,\dots\dots,r-k.$$

In the case of a regular manifold, $k=0$ and $A_0=\alpha_0=\sigma_{n-r}$ is the order d of the manifold. Then equations (9) read thus:

$$\begin{aligned}
d&=\alpha_0,\\
0&=\alpha_0\sigma_1+\alpha_1,\\
0&=\alpha_0\sigma_2+\alpha_1\sigma_1+\alpha_2,\\
&\dots\dots\dots\dots\dots\dots\dots\dots,\\
0&=\alpha_0\sigma_j+\alpha_1\sigma_{j-1}+\alpha_2\sigma_{j-2}+\dots\dots+\alpha_{j-1}\sigma_1+\alpha_j.
\end{aligned}$$

Multiplying these in order, beginning with the last, by $1, -P_1, P_2, -P_3, \dots\dots, (-1)^j P_j$, where the P's are the complete symmetric polynomials formed from the given orders, and recalling that

$$(10)\qquad \sigma_l-P_1\sigma_{l-1}+P_2\sigma_{l-2}-P_3\sigma_{l-3}+\dots\dots+(-1)^l P_l=0,$$

we find that $\alpha_j=(-1)^j d P_j$.

(11) *The index numbers $\alpha_k, \dots\dots, \alpha_r$ of the complete manifold M_r in S_n of excess k are determined in terms of the given orders and the earlier index numbers by the equations* (9); *those of a regular manifold are $\alpha_j=(-1)^j d P_j$, $j=0,\dots\dots,r$, where d is the product, P_j the complete symmetric polynomial of order j, formed from the orders of the $n-r$ spreads in S_n which cut out M_r; those of an ordinary manifold M_{n-1} in S_n of order d are $\alpha_j=(-1)^j d^{j+1}$, $j=0,\dots\dots,n-1$.*

In particular, if M_r is a linear manifold S in S_n, it is regular, being cut out by $n-r$ ordinary linear spreads, and P_j reduces to the number of terms in the complete symmetric polynomial of order j in $n-r$ variables, which is known to be $\binom{n-r+j-1}{j}$, whence

(12) *The index numbers of an S_r in S_n are*

$$\alpha_j=(-1)^j\binom{n-r+j-1}{j},\qquad j=0,1,\dots\dots,r.$$

5. If M_r in S_n has index numbers α, then M_r in an S_{n+1} containing S_n has index numbers α', where

$$(13) \qquad \alpha'_j = \alpha_j - \alpha_{j-1} + \alpha_{j-2} - \alpha_{j-3} + \dots + (-1)^j \alpha_0 .$$

This is true of α'_0, the order of M_r; let us assume it for $\alpha'_1, \dots, \alpha'_{r-1}$ and prove it for α'_r. If in S_n α_r is determined from $O_r = \sigma_n - A_r$, in S_{n+1} α'_r is determined by an additional linear spread, the S_n itself, O_r being the same as before. Hence $O_r = 1 \cdot \sigma_n - A'_r$, where in A'_r $\sigma'_k = \sigma_k + \sigma_{k-1}$. Thus

$$A_r = A'_r = \alpha'_0(\sigma_r + \sigma_{r-1}) + \alpha'_1(\sigma_{r-1} + \sigma_{r-2}) + \dots + \alpha'_{r-1}(\sigma_1 + 1) + \alpha'_r,$$

or

$$A_r = \alpha'_0\sigma_r + (\alpha'_0 + \alpha'_1)\sigma_{r-1} + (\alpha'_1 + \alpha'_2)\sigma_{r-2} + \dots + (\alpha'_{r-2} + \alpha'_{r-1})\sigma_1 + (\alpha'_{r-1} + \alpha'_r).$$

Assuming (13) up to $j = r$, this determines α'_r, which is of the same form since $\alpha'_k + \alpha'_{k-1} = \alpha_k$. By repeated use of (13) we find that

(14) *If the index numbers of M_r in S_n are $\alpha_0, \dots, \alpha_r$, the index numbers of the same M_r in S_{n+p} are*

$$\alpha_j^{(p)} = \alpha_j - \binom{p}{1}\alpha_{j-1} + \binom{p+1}{2}\alpha_{j-2} - \binom{p+2}{3}\alpha_{j-3} + \dots + (-1)^j\binom{p+j-1}{j}\alpha_0, \qquad j = 0, 1, \dots, r.$$

This, applied to an S_r in S_n, yields through (12) the relation

$$(15) \quad \binom{b}{0}\binom{a+c}{a} + \binom{b+1}{1}\binom{a+c-1}{a} + \binom{b+2}{2}\binom{a+c-2}{a} + \dots + \binom{b+c}{c}\binom{a}{a} = \binom{a+b+c+1}{c}.$$

6. Let $M_r(\alpha)$ in S_n be cut by the spread F of order q in $M_{r-1}(\alpha')$. Then

$$(16) \quad \begin{cases} \alpha'_0 = q\alpha_0; \quad \text{let us assume} \\ \alpha'_1 = q\alpha_1 - q^2\alpha_0, \\ \alpha'_2 = q\alpha_2 - q^2\alpha_1 + q^3\alpha_0, \\ \dots\dots\dots\dots\dots\dots\dots\dots, \\ \alpha'_{r-2} = q\alpha_{r-2} - q^2\alpha_{r-3} + q^3\alpha_{r-4} - \dots + (-1)^{r-2}q^{r-1}\alpha_0; \text{ and prove that} \\ \alpha'_{r-1} = q\alpha_{r-1} - q^2\alpha_{r-2} + q^3\alpha_{r-3} - \dots + (-1)^{r-1}q^r\alpha_0 . \end{cases}$$

Let spreads of orders $\lambda_1, \dots, \lambda_{n-1}$ on M_r meet outside M_r in a curve C whose order (from a linear section) is $O_{r-1} = \sigma_{n-1} - A_{r-1}$, the σ's being formed from $\lambda_1, \dots, \lambda_{n-1}$. These $n-1$ spreads together with F make up n spreads on M_{r-1} which meet again in O'_{r-1} points outside M_{r-1}, the points where F meets C. Hence $O'_{r-1} = q\,O_{r-1} = \sigma'_n - A'_{r-1}$, A'_{r-1} being formed from the α''s and σ''s, which in turn are formed from $\lambda_1, \dots, \lambda_{n-1}, q$. Hence $A'_{r-1} = q\,A_{r-1}$ and α'_{r-1} is determined by

$$\alpha_0'(\sigma_r + q\,\sigma_{r-1}) + \alpha_1'(\sigma_{r-1} + q\,\sigma_{r-2}) + \ldots\ldots + \alpha_{r-2}'(\sigma_1 + q) + \alpha_{r-1}' = q\,(\alpha_0\sigma_r + \ldots\ldots + \alpha_{r-1}),$$

or

$$\alpha_0'\sigma_r + (\alpha_1' + q\,\alpha_0')\,\sigma_{r-1} + \ldots\ldots + (\alpha_{r-1}' + q\,\alpha_{r-2}') = q\,(\alpha_0\sigma_r + \ldots\ldots + \alpha_{r-1}).$$

Assuming (16) true throughout, this equation is satisfied since then $\alpha_j' + q\,\alpha_{j-1}' = q\alpha_j$, $j = 0, 1, \ldots\ldots, r-1$. Hence

(17) *If $M_r(\alpha)$ in S_n is cut by a spread of order q in an M_{r-1}, the index numbers α' of M_{r-1} in S_n are given by* (16) *in terms of q and the α's.*

For $q = 1$ we have again the formula (13). If M_r is a regular manifold, so also is M_{r-1}, and (16) merely expresses that $P_k' = q^k + q^{k-1}P_1 + \ldots\ldots + P_k$, where P_i' is formed from $\lambda_1, \ldots\ldots, \lambda_{n-1}$, q and P_i from $\lambda_1, \ldots\ldots, \lambda_{n-1}$ only.

7. The theorem (17) can be generalized as follows:

(18) *If $M_r(\alpha)$ and $M_s(\beta)$ in S_n meet in an $M_{r+s-n}(\gamma)$*, then $\gamma_k = \sum_{i=0}^{k} \alpha_i \beta_{k-i}$.

Let us prove this provisionally for the case where one of the manifolds, say $M_s(\beta)$, is regular, being determined by $n-s$ spreads of orders $p, q, \ldots\ldots$ Then M_{r+s-n} is the meet of $M_r(\alpha)$ and these spreads in order, and the index numbers of the successive sections according to (17) are

$$\begin{array}{lll}
p\,\alpha_0, & p\,q\,\alpha_0, & \ldots\ldots, \\
p\,\alpha_1 - p^2\alpha_0, & p\,q\,\alpha_1 - p\,q\,(p+q)\,\alpha_0, & \ldots\ldots, \\
p\,\alpha_2 - p^2\alpha_1 + p^3\alpha_0, & p\,q\,\alpha_2 - p\,q\,(p+q)\,\alpha_1 + p\,q\,(p^2+q^2+p\,q)\,\alpha_0, & \ldots\ldots, \\
\ldots\ldots\ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, & \ldots\ldots
\end{array}$$

Substituting the index numbers of $M_s(\beta)$ as given by (11), we have the desired formula. A more general statement is:

(19) *If in S_n b spreads $M_{r_1}(\alpha^{(1)}), M_{r_2}(\alpha^{(2)}), \ldots\ldots, M_{r_b}(\alpha^{(b)})$ meet in an $M_{\Sigma r_i - n(b-1)}$ with index numbers β, then $\beta_k = \Sigma\,\alpha_{i_1}^{(1)}\alpha_{i_2}^{(2)}\ldots\ldots\alpha_{i_b}^{(b)}$, where $i_1 + i_2 + \ldots\ldots + i_b = k$.*

In (17), (18), (19) we have generalizations of (11) relating to regular manifolds. If in (19) the dimensions $r_1, \ldots\ldots, r_b$ are all equal to $n-1$, $M(\beta)$ is a regular M_{n-b}.

8. Given $M_r(\alpha)$ in S_n; let α' be the index numbers of M_r doubled, *i. e.* $[M_r]^2$. A section of M_r by an S_{n-r} is α_0 points, which are doubled if M_r is doubled, and account for $2^{n-r}\alpha_0$ intersections, whence $\alpha_0' = 2^{n-r}\alpha_0$. Let us assume that $\alpha_k' = 2^{n-r+k}\alpha_k$ for $k = 0, 1, \ldots\ldots, r-1$ and prove that $\alpha_r' = 2^n\alpha_r$. Let M_r be determined to within certain outside points by $f_1, \ldots\ldots, f_n$, of orders $\lambda_1, \ldots\ldots, \lambda_n$, and also by $g_1, \ldots\ldots, g_n$, of orders $\mu_1, \ldots\ldots, \mu_n$. Then $[M_r]^2$ is determined by $f_1 \cdot g_1, \ldots\ldots, f_n \cdot g_n$ to within O_r' outside points, or

$$\alpha_r' = \prod_1^n(\lambda_i + \mu_i) - \alpha_0'\sigma_r(\lambda_i + \mu_i) - \alpha_1'\sigma_{r-1}(\lambda_i + \mu_i) - \ldots\ldots - \alpha_{r-1}'\sigma_1(\lambda_i + \mu_i) - O_r'.$$

The points O'_r arise from any k spreads f and $n-k$ complementary spreads g, whence

$$O'_r = \Sigma\,[\lambda_1\,\lambda_2 \dots \lambda_k\,\mu_{k+1} \dots \mu_n - \alpha_0\,\sigma_r - \alpha_1\,\sigma_{r-1} - \dots - \alpha_{r-1}\,\sigma_1 - \alpha_r],$$

where Σ refers to the 2^n choices of $\lambda_1, \dots, \lambda_k, \mu_{k+1}, \dots, \mu_n$. Collecting the terms in O'_r, we find that

$$O'_r = \prod_1^n (\lambda_i + \mu_i) - 2^{n-r}\alpha_0\,\sigma_r\,(\lambda_i + \mu_i) - 2^{n-r+1}\alpha_1\,\sigma_{r-1}\,(\lambda_i + \mu_i) - \dots$$
$$- 2^{n-1}\alpha_{r-1}\sigma_1\,(\lambda_i + \mu_i) - 2^n\alpha_r.$$

Substituting this value in α'_r and using the assumed values of $\alpha'_0, \dots, \alpha'_{r-1}$, we find $\alpha'_r = 2^n\alpha_r$, whence

(20) *If the index numbers of* M_r *in* S_n *are* α, *the index numbers of* $[M_r]^2$ *are* $\alpha'_j = 2^{n-r+j}\alpha_j$, $j = 0, 1, \dots, r$.

Using an additional set of spreads $h_1, \dots, h_n$ on M_r of orders $\nu_1, \dots, \nu_n$, and a similar argument, we find the index numbers of $[M_r]^3$, etc. The result is as follows:

(21) *The index numbers of an l-fold* M_r *in* S_n *in terms of those of the simple* M_r *are* $\alpha'_j = l^{n-r+j}\alpha_j$.

9. Let $M_r(\alpha)$ in S_n be contained simply on spreads of orders $\lambda_1, \dots, \lambda_n$. We ask for the number O_r of points outside of M_r common to n spreads $f_1, \dots, f_n$ of orders $l_1, \dots, l_n$, which contain M_r $k_1, \dots, k_n$ times respectively. Consider the n degenerate spreads,

$$u_1^{(1)} \cdot u_2^{(1)} \cdot \dots \cdot u_{k_1}^{(1)} \cdot P^{(l_1 - k_1\lambda_1)}, \dots, u_1^{(n)} \cdot u_2^{(n)} \cdot \dots \cdot u_{k_n}^{(n)}\, P^{(l_n - k_n\lambda_n)},$$

where $u_i^{(h)}$ is a spread of order λ_h containing M_r simply, and $P^{(l_i - k_i\lambda_i)}$ is a spread of order $l_i - k_i\lambda_i$ in general position with regard to M_r. These degenerate spreads satisfy the conditions of the problem, and from them O_r can be calculated. We find that

$$O_r = \prod_1^n (l_i - k_i\lambda_i) + \Sigma \prod_1^{n-1} (l_i - k_i\lambda_i)\,k_n\lambda_n + \Sigma \prod_1^{n-2} (l_i - k_i\lambda_i)\,\lambda_{n-1}\lambda_n k_{n-1} k_n$$
$$+ \dots + \Sigma \prod_1^{r} (l_i - k_i\lambda_i)\,(\lambda_{r+1}\lambda_{r+2} \dots \lambda_n - \alpha_0)\,k_{r+1}k_{r+2} \dots k_n$$
$$+ \Sigma \prod_1^{r-1} (l_i - k_i\lambda_i)\,(\lambda_r\lambda_{r+1} \dots \lambda_n - \alpha_0\,\sigma_1 - \alpha_1)\,k_r k_{r+1} \dots k_n$$
$$+ \Sigma \prod_1^{r-2} (l_i - k_i\lambda_i)\,(\lambda_{r-1}\lambda_r \dots \lambda_n - \alpha_0\sigma_2 - \alpha_1\sigma_1 - \alpha_2)\,k_{r-1}k_r \dots k_n$$
$$+ \dots + (\lambda_1\lambda_2 \dots \lambda_n - \alpha_0\,\sigma_r - \alpha_1\sigma_{r-1} \dots - \alpha_r)\,k_1 k_2 \dots k_n.$$

To divide this by $k_1 \cdot k_2 \cdot \dots \cdot k_n$ amounts to replacing l_i by $\frac{l_i}{k_i}$ and dropping the other k's. Then O_r takes the form $\overline{O}_r$, which is obtained from n spreads

$v^{(i)}\cdot P\left(\frac{l_i}{k_i}-\lambda_i\right)$, $v^{(i)}$ being of order λ_i, *i. e.*, the number $\overline{O}_r$ obtained from n spreads of orders $\frac{l_i}{k_i}$ containing M_r simply. Hence

(22) *The number of points not on $M_r(\alpha)$ in S_n and common to spreads $f_1, \dots, f_n$ of orders $l_1, \dots, l_n$, which contain M_r $k_1, \dots, k_n$ times respectively, is given by the formula*

$$O_r = [\sigma_n - \alpha_0\sigma_r - \alpha_1\sigma_{r-1} - \alpha_2\sigma_{\;-2} - \dots - \alpha_r]\, k_1 k_2 \dots k_n:$$

where the σ's refer to $\frac{l_1}{k_1}, \dots, \frac{l_n}{k_n}$.

If the k's are all equal, this furnishes the same result as (21).

10. In S_n let M'_r and M''_s be two *non-incident* manifolds with index numbers α' and α'' respectively, where $s \overline{<} r$ and $r+s<n$. Let the index numbers of the two manifolds considered as a whole be α. From their sections we see that $\alpha_j = \alpha'_j$ for $j=0, 1, \dots, r-s-1$. In determining α_{r-s}, we find that O_{r-s} is O'_{r-s} diminished by α''_0, the order of M''_s, whence $\alpha_{r-s} = \alpha'_{r-s} + \alpha''_0$. Let us assume, then, that $\alpha_j = \alpha'_j + \alpha''_{j-(r-s)}$ for $j = 0, 1, \dots, r-1$, and prove that the formula holds when $j=r$ also. If to within certain outside points M'_r is determined by spreads $f_1, \dots, f_n$ of orders $\lambda_1, \dots, \lambda_n$, and M''_s by spreads $g_1, \dots, g_n$ of orders $\mu_1, \dots, \mu_n$, then $M'_r \cdot M''_s$ is determined by spreads $f_1\cdot g_1, \dots, f_n \cdot g_n$ of orders $\lambda_1+\mu_1, \dots, \lambda_n+\mu_n$ to within O_r outside points. These points arise from the intersection of the meet of $f_1, \dots, f_k$ (residual to M'_r if $k \overline{>} n-r$) and the meet of $g_{k+1}, \dots, g_n$ (residual to M''_s if $n-k \overline{>} n-s$). The orders of these residual intersections are determined by a proper section, and we find that

$$\begin{aligned} O_r = \Sigma\, &\{\lambda_1\lambda_2 \dots \lambda_k - \alpha'_0\sigma_{k-(n-r)} - \alpha'_1\sigma_{k-1-(n-r)} - \dots - \alpha'_{k-(n-r)}\} \\ &\cdot\{\mu_{k+1}\mu_{k+2} \dots \mu_n - \alpha''_0\,\sigma_{s-k} - \alpha''_1\,\sigma_{s-1-k} - \dots - \alpha''_{s-k}\}, \end{aligned}$$

the Σ referring to all possible complementary choices of n λ's and μ's, and the σ's in a brace referring to the quantities appearing in the first product of the brace. From this value of O_r we can find α_r. If we suppose that $\alpha_r = \alpha'_r + \alpha''_s$, we have only to verify that O_r above and

$$\begin{aligned} O = {} & \sigma_n(\lambda_i+\mu_i) - \alpha'_0\sigma_r(\lambda_i+\mu_i) - \dots - \alpha'_{r-s-1}\sigma_{s+1}(\lambda_i+\mu_i) \\ & - (\alpha'_{r-s}+\alpha''_0)\,\sigma_s(\lambda_i+\mu_i) - \dots - (\alpha'_{r-1}+\alpha''_{s-1})\,\sigma_1(\lambda_i+\mu_i) - (\alpha'_r+\alpha''_s) \end{aligned}$$

are the same. In the first value the terms free of α' and α'' are $\sigma_n(\lambda_i+\mu_i)$, as also in the second. There are no terms in the first expression containing both an α' and an α'', since $r+s<n$. In each expression an α' or α'' is multiplied by all possible terms of a definite simultaneous degree in λ and μ, whence

their coefficients must be the same. This identifies the two expressions, and the assumed formula for α_j is generally true.

(23) *If two non-incident spreads* M'_r *and* M''_s *in* S_n $(s \overline{<} r,\ r+s<n)$ *have index numbers* α' *and* α'' *respectively, the two together constitute a manifold which has the index numbers* $\alpha_j = \alpha'_j + \alpha''_{j-(r-s)}$, $j = 0, 1, \ldots, r$.

The generalization of (23) to the case of any number of manifolds, no two of which have common points, is obvious. The argument used does not apply (nor is the result true) in the case where M'_r and M''_s have common points, say a common M'''_t, *unless* M'''_t be doubled. This is due to the nature of the spreads used above to determine O_r.

11. Given a curve $M_1(\alpha)$ in S_n, $n-1$ spreads of orders $\lambda_1, \ldots, \lambda_n$ on it meet in a residual curve $M_1(\beta)$ which cuts $M_1(\alpha)$ in $M_0(\vartheta)$ points. From a section $\alpha_0+\beta_0=\sigma_{n-1}$. A further spread of order l on $M_1(\alpha)$ cuts $M_1(\beta)$ in $\beta_0 l - \vartheta_0$ points outside $M_1(\alpha)$. This number is given by $O_1 = l\,\sigma_{n-1} - \alpha_0(\sigma_1+l) - \alpha_1$. Equating the two numbers, we find from the coefficients of l that $\alpha_0+\beta_0=\sigma_{n-1}$ and $\vartheta_0 = \alpha_0\sigma_1+\alpha_1$. The two curves are mutually related, since they constitute a regular curve, whence also $\vartheta_0 = \beta_0\sigma_1+\beta_1$. The index numbers of the two according to (11) are σ_{n-1} and $-\sigma_1\sigma_{n-1}$ or $\alpha_0+\beta_0$ and $-(\alpha_0+\beta_0)\sigma_1 = \alpha_1+\beta_1-2\vartheta_0$. That these are the index numbers of any two curves $M_1(\alpha)$ and $M_1(\beta)$ with ϑ_0 common points can be proved directly. Let $n-1$ spreads on the two meet again in $M_1(\gamma)$, which cuts $M_1(\alpha)$ in η_0 points and $M_1(\beta)$ in ζ_0 points. If the index numbers of the pair $M_1(\alpha)$, $M_1(\beta)$ are $\alpha_0+\beta_0$, $\alpha_1+\beta_1+x_{\alpha\beta}$, and similarly for the other pairs, then we have from the above, regarding each curve in turn as residual, the following relations:

$$\begin{aligned}
(\alpha_0+\beta_0)\,\sigma_1+\alpha_1+\beta_1+x_{\alpha\beta} &= \eta_0+\zeta_0 = \gamma_0\sigma_1+\gamma_1,\\
(\beta_0+\gamma_0)\,\sigma_1+\beta_1+\gamma_1+x_{\beta\gamma} &= \zeta_0+\vartheta_0 = \alpha_0\sigma_1+\alpha_1,\\
(\gamma_0+\alpha_0)\,\sigma_1+\gamma_1+\alpha_1+x_{\alpha\gamma} &= \vartheta_0+\eta_0 = \beta_0\sigma_1+\beta_1.
\end{aligned}$$

From the first and the sum of the last two we find that $x_{\alpha\beta} = -2\vartheta_0$. Hence

(24) *If* $n-1$ *spreads of orders* $\lambda_1, \ldots, \lambda_n$ *on* $M_1(\alpha)$ *in* S_n *meet in a residual* $M_1(\beta)$ *which has* $M_0(\vartheta)$ *points in common with* $M_1(\alpha)$, *then the index numbers* β_0, β_1, ϑ_0 *are determined from the equations*

$$\alpha_0+\beta_0=\sigma_{n-1}, \quad \alpha_0\sigma_1+\alpha_1=\vartheta_0=\beta_0\sigma_1+\beta_1.$$

The index numbers of an $M_1(\alpha)$ *and* $M_1(\beta)$ *with common* $M_0(\vartheta)$ *are* $\alpha_0+\beta_0$, $\alpha_1+\beta_1-2\vartheta_0$.

Given a curve $M_1(\alpha)$ in S_3 of order ν, genus π, and rank γ (class of the projection) with d apparent double points, let the cones on $M_1(\alpha)$ with vertices at P_1 and P_2 such that $\overline{P_1P_2}$ is skew to $M_1(\alpha)$ meet in a residual curve $M_1(\beta)$

of order $\nu^2 - \nu$ which meets $M_1(\alpha)$ in ϑ_0 points. These ϑ_0 points arise (a) from the points where a double generator on P_1 or P_2 meets $M_1(\alpha)$, *i. e.*, $4\,d$ points, and (b) from the γ points on $M_1(\alpha)$ of contact of planes on $\overline{P_1 P_2}$. Thus $\vartheta_0 = 4\,d + \gamma$, where $\gamma = 2\,(\nu - 1 + \pi)$ and $d = \frac{1}{2}\,(\nu - 1)\,(\nu - 2) - \pi$. But $\vartheta_0 = \alpha_0 \sigma_1 + \alpha_1 = \nu\,(2\,\nu) + \alpha_1$, whence $\alpha_1 = -2\,\nu - 2\,(\nu - 1 + \pi)$. If also $M_1(\alpha)$ has δ actual double points, $M_1(\beta)$ has the same double points, each of which counts for two points in ϑ_0, whence $\vartheta_0 = 4\,(d + \delta) + \gamma - 2\,\delta$, and $\alpha_1 = -2\,\nu - 2\,(\nu - 1 + \pi + \delta)$. Hence

(25) *The index numbers of a curve in* S_3 *of order* ν, *genus* π, *and rank* γ, *with* δ *actual double points, are*

$$\alpha_0 = \nu, \quad \alpha_1 = -2\,\nu - 2\,(\nu - 1 + \pi + \delta) = -2\,\nu - \gamma - 2\,\delta;$$

i. e., they are the same as those of ν *lines with* $\nu - 1 + \pi + \delta$ *intersections.*

Using these index numbers and (22), we find that

(26) *Three surfaces of orders* n_1, n_2, n_3 *which contain* k_1, k_2, k_3 *times respectively a space curve of order* ν *and rank* γ, *with* δ *actual double points, meet again in*

$$n_1 n_2 n_3 - \nu\,(n_1 k_2 k_3 + n_2 k_3 k_1 + n_3 k_1 k_2) + k_1 k_2 k_3 (2\,\nu + \gamma + 2\,\delta)$$

points.[*]

The index numbers of a curve with a higher singularity do not depend on the order of the singularity alone. For example, if the tangents at a triple point lie in a plane, the curve behaves at that point like three lines in a plane with three intersections, otherwise like two lines meeting a third with two intersections.

(27) *The index numbers in* S_n *of a curve of order* ν *and genus* π, *with* δ *actual double points, are* ν, $-\nu\,(n - 1) - 2\,(\nu - 1 + \pi + \delta)$.

This is certainly true according to (24), if the curve in S_n, as in S_3, can be replaced by ν lines with $(\nu - 1 + \pi + \delta)$ intersections. It merely requires that the second index number shall increase by ν, if the curve is projected into the space of next lower dimension. This requirement is satisfied on projection from S_3 to S_2, since the index numbers in S_2 are ν, $-\nu^2$ [see (11)]. It is satisfied also on projection from an $S_{k+\pi}$ to an $S_{k+\pi-1}$, when the curve itself lies in an S_k [see (13)]. We shall assume provisionally that the requirement is satisfied for intermediate projections.

12. It is clear that the formula (7) for the order of a restricted system will in most applications be a rather complicated expression. In order to

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[*] Cf. Pascal-Schepp, *Repertorium,* Vol. II (1902), p. 212.

simplify it, certain numerical relations among the binomial coefficients are given here. Most and possibly all of these are known; but for lack of definite references proofs of the more complicated relations are indicated.

1°. $\binom{a}{b}=\binom{a}{a-b}$.

2°. $\binom{a}{b}\binom{a-b}{c}=\binom{a}{b+c}\binom{b+c}{c}=\binom{a}{c}\binom{a-c}{b}$.

3°. $\binom{a}{b}=\binom{a-1}{b}+\binom{a-1}{b-1}$.

4°. $\binom{a+1}{b}=\binom{a}{b}+\binom{a-1}{b-1}+\binom{a-2}{b-2}+\dots\dots+\binom{a-b}{0}$.

This is proved by repeated use of 3°.

5°. $$\binom{a+c}{b}=\binom{c-1}{0}\binom{a}{b}+\binom{c}{1}\binom{a-1}{b-1}+\binom{c+1}{2}\binom{a-2}{b-2}+\dots\dots$$
$$+\binom{c+b-1}{b}\binom{a-b}{0}.$$

This is proved by repeated use of 4° [see (15)].

6°. $$\binom{a+c}{b}=\binom{c}{0}\binom{a}{b}+\binom{c}{1}\binom{a}{b-1}+\binom{c}{2}\binom{a}{b-2}+\dots\dots$$
$$+\binom{c}{c}\binom{a}{b-c}.$$

This is proved by repeated use of 3°.

7°. $\binom{a}{b}=\binom{a+1}{b}-\binom{a+1}{b-1}+\binom{a+1}{b-2}-\dots\dots+(-1)^b\binom{a+1}{0}$.

This follows immediately from 3°.

8°. $$\binom{a}{b}=\binom{c-1}{0}\binom{a+c}{b}-\binom{c}{1}\binom{a+c}{b-1}+\binom{c+1}{2}\binom{a+c}{b-2}-\dots\dots$$
$$+(-1)^b\binom{c+b-1}{b}\binom{a+c}{0}.$$

This is proved by using 7° repeatedly.

9°. $$\binom{a}{0}\binom{a}{b}s^{a-b}-\binom{a}{1}\binom{a-1}{b}s^{a-b-1}t+\binom{a}{2}\binom{a-2}{b}s^{a-b-2}t^2-\dots\dots$$
$$+(-1)^{a-b}\binom{a}{a-b}\binom{b}{b}t^b=\binom{a}{b}(s-t)^{a-b}.$$

The factor $\binom{a}{b}$ can be removed from each term by using 2°.

10°. $$\binom{a}{0}\binom{a+c}{b}-\binom{a}{1}\binom{a+c-1}{b}+\binom{a}{2}\binom{a+c-2}{b}-\dots\dots$$
$$+(-1)^{a+c-b}\binom{a}{a+c-b}\binom{b}{b}=0,\text{ if } b<a.$$

Reduce the upper numbers in $\binom{a+c}{b}$, $\binom{a+c-1}{b}$, ... by means of 6° and apply 9° for $s=t=1$.

11°.
$$\binom{a-1}{0}\binom{a+c}{b}s^b-\binom{a}{1}\binom{a+c}{b-1}s^{b-1}t+\binom{a+1}{2}\binom{a+c}{b-2}s^{b-2}t^2$$
$$-\ldots\ldots+(-1)^b\binom{a+b-1}{b}\binom{a+c}{0}t^b$$
$$=\binom{c-b}{0}\binom{a+c}{b}(s-t)^b+\binom{c-b+1}{1}\binom{a+c}{b-1}(s-t)^{b-1}t$$
$$+\binom{c-b+2}{2}\binom{a+c}{b-2}(s-t)^{b-2}t^2+\ldots\ldots+\binom{c}{b}\binom{a+c}{0}t^b$$
$$=\binom{a-1}{0}\binom{c}{b}s^b+\binom{a}{1}\binom{c-1}{b-1}s^{b-1}(s-t)$$
$$+\binom{a+1}{2}\binom{c+2}{b+2}s^{b-2}(s-t)^2+\ldots\ldots+\binom{a+b-1}{b}\binom{c-b}{0}(s-t)^b.$$

In the first expression raise the upper number of the first factor of each term by $c-b+1$, according to 8°. Each term of the result can then be modified by 2°. Collecting the coefficients of $\binom{c-b}{0}$, $\binom{c-b+1}{1}$, $\binom{c-b+2}{2}$, ..., and using the expansion of $(s-t)^i$, the second expression is obtained. If in this $a+c$ be reduced by a or more, according to 5°, and the coefficients of $\binom{c}{b}$, $\binom{c-1}{b-1}$, $\binom{c-2}{b-2}$, ... be collected, the third expression is obtained.

13. If point conics in a plane be mapped on the points of an S_5, the ∞^2 repeated lines of the plane are mapped on the points of the Veronese surface, F_2^4. The point conics apolar to a given line conic are mapped on an S_4 in S_5, those apolar to two given line conics on an S_3 in S_5. The latter system contains the squares of the common lines of the two line conics, whence α_0 of F_2^4 is 4. A section of F_2^4 by an S_4 is a quartic curve necessarily rational, since it lies in S_4 or since it is the map of the lines of a line conic. Using (27) for $\nu=n=4$, $\pi=\delta=0$, we find that $\alpha_1=-18$. The point conics which, in line form, are apolar to a given point conic map on a quadric which contains F_2^4. Since there is only one proper conic apolar to five given point conics, five such quadrics on F_2^4 meet in one outside point, whence to determine α_2 we have the equation $1=2^5-\alpha_0\binom{5}{2}2^2-\alpha_1\binom{5}{1}2-\alpha_2$ or $\alpha_2=51$.

(28) *The index numbers of the Veronese surface* F_2^4 *in* S_5 *are* 4, —18, 51.

If $(ax)^2=(a'x)^2=\ldots\ldots$ be a variable point conic, and $(bx)^2=(b'x)^2=\ldots\ldots$ be a fixed point conic, then in the discriminant equation of the pencil,

$$(a a' a'')^2 + 3\lambda\,(a a' b)^2 + 3\lambda^2\,(a b b')^2 + \lambda^3\,(b b' b'')^2 = 0,$$

$(a a' a'')^2 = 0$ is a cubic spread in S_5 containing F_2^4 doubly, $(a a' b)^2 = 0$ is a quadric spread containing F_2^4 simply, $(a b b')^2 = 0$ is an S_4, while $(b b' b'')^2$ is a constant. Hence the discriminant of this cubic equation—the tact-invariant of the two conics—is in S_5 a sextic spread containing F_2^4 doubly. According to (20) the index numbers of F_2^4 doubled, are $2^3 \cdot 4$, $-2^4 \cdot 18$, $2^5 \cdot 51$, and five such sextic spreads will meet outside F_2^4 in

$$O_2 = 6^5 - \binom{5}{2} 6^2 \cdot 2^3 \cdot 4 + \binom{5}{1} 6 \cdot 2^4 \cdot 18 - 2^5 \cdot 51 = 2^5 \cdot 102 = 3264$$

points, whence we have the well-known theorem:*

(29) *There are* 3264 *proper conics which touch five given proper conics.*

14. Let us ask how many proper conics touch a general rational plane quintic five times. Conics cut the quintic in sets of ten points determined by binary ten-ics apolar to five binary ten-ics. We want, then, the number of squared quintics apolar to the five ten-ics exclusive of the ∞^2 quintics determined by line sections whose squares obviously satisfy the apolarity conditions. Map binary quintics upon the points of an S_5, and the line sections will map on points of an S_2. The apolarity condition of the squared quintic and one of the five ten-ics represents in S_5 a quadric which contains the S_2. The index numbers of the S_2 in S_5 are 1, —3, 6, whence five quadrics on S_2 meet outside of S_2 in $O_2 = 2^5 - \binom{5}{2} 2^2 \cdot 1 + \binom{5}{1} 2 \cdot 3 - 6 = 16$ points, or

(30) *There are* 16 *proper conics which touch a general rational plane quintic curve five times.*†

15. A plane curve of order r, f^r, is determined by $\frac{1}{2} r(r+3)$ constants. If it degenerates into an f^s and an f^{r-s}, $s < r$, the two contain only $\lambda + \mu$ constants, where $\lambda = \frac{1}{2} s(s+3)$ and $\mu = \frac{1}{2}(r-s)(r-s+3)$, whence to degenerate thus is $s(r-s)$ conditions. Given then a linear system of $\infty^{s(r-s)}$ f^r's, how many members of the system degenerate into an f^s and an f^{r-s}. The system is apolar to $\lambda + \mu$ linearly independent curves of class r; *i. e.*, the coefficients of its members satisfy $\lambda + \mu$ apolarity conditions. Map the curves f^s upon the points of an S_λ, the curves f^{r-s} upon the points of an S_μ which lies skew to S_λ in an $S_{\lambda+\mu+1}$. Then a product $f^s \cdot f^{r-s}$ can be mapped by any point of the line joining the maps of f^s and of f^{r-s}. Taking a section by an $S_{\lambda+\mu}$, such a line or such a product is represented by a point. Conversely, a point in $S_{\lambda+\mu}$

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* Pascal–Schepp, *Repertorium*, Vol. II (1902), p. 433.

† Cf. F. Morley, "The Contact Conics of the Plane Quintic Curve," *Johns Hopkins University Circular* (1912), No. 2.

represents such a product unless it lies in the section $S_{\lambda-1}$ or the section $S_{\mu-1}$, in which case f^{r-s} or f^s respectively vanishes identically. The apolarity relations furnish $\lambda+\mu$ quadrics in $S_{\lambda+\mu}$ on $S_{\lambda-1}$ and $S_{\mu-1}$. The index numbers of $S_{\lambda-1}$ in $S_{\lambda+\mu}$ are $1, -\binom{\mu+1}{1}, \binom{\mu+2}{2}, \ldots\ldots, (-1)^{\lambda-1}\binom{\lambda+\mu-1}{\lambda-1}$; those of $S_{\mu-1}$ are $1, -\binom{\lambda+1}{1}, \binom{\lambda+2}{2}, \ldots\ldots, (-1)^{\mu-1}\binom{\lambda+\mu-1}{\mu-1}$. Since the $S_{\lambda-1}$ and $S_{\mu-1}$ have no common points, according to (23) their index numbers are additive and the $\lambda+\mu$ quadrics meet outside of $S_{\lambda-1}$ and $S_{\mu-1}$ in

$$O = 2^{\lambda+\mu} - \Big[\binom{\lambda+\mu}{\lambda-1}2^{\lambda-1} - \binom{\lambda+\mu}{\lambda-2}2^{\lambda-2}\cdot\binom{\mu+1}{1} + \binom{\lambda+\mu}{\lambda-3}2^{\lambda-3}\cdot\binom{\mu+2}{2}$$
$$- \ldots\ldots + (-1)^{\lambda}\binom{\lambda+\mu}{0}2^{0}\cdot\binom{\mu+\lambda-1}{\lambda-1}\Big]$$
$$-\Big[\binom{\lambda+\mu}{\mu-1}2^{\mu-1} - \binom{\lambda+\mu}{\mu-2}2^{\mu-2}\cdot\binom{\lambda+1}{1} + \binom{\lambda+\mu}{\mu-3}2^{\mu-3}\cdot\binom{\lambda+2}{2}$$
$$- \ldots\ldots + (-1)^{\mu}\binom{\lambda+\mu}{0}2^{0}\cdot\binom{\lambda+\mu-1}{\mu-1}\Big].$$

Apply 11° to the brackets and they become respectively

$$\Big[\binom{\lambda+\mu}{\mu-1}+\binom{\lambda+\mu}{\lambda-2}+\ldots\ldots+\binom{\lambda+\mu}{0}\Big] \text{ and } \Big[\binom{\lambda+\mu}{\mu-1}+\binom{\lambda+\mu}{\mu-2}+\ldots\ldots+\binom{\lambda+\mu}{0}\Big].$$

The sum of the two is $(1+1)^{\lambda+\mu} - \binom{\lambda+\mu}{\lambda}$, whence $O = \binom{\lambda+\mu}{\lambda}$.

(31) *In a linear system of* $\infty^{s(r-s)}$ *plane curves of order r there are* $\binom{\lambda+\mu}{\lambda}$ *which break up into a curve of order s and a curve of order $r-s$,* $\lambda = \tfrac{1}{2}s(s+3)$, $\mu = \tfrac{1}{2}(r-s)(r-s+3)$.

The same result is obtained at once by the methods of Schubert, if the $\lambda+\mu$ curves of class r be taken as r-fold points.

16. Given a rational curve of order ν in S_k, how many S_{k-1}'s meet it λ times in μ coincident points, $\lambda(\mu-1) = k$? The S_{k-1}'s determine on S_k $k+1$ linearly independent binary forms f^{ν} which are apolar to $\nu-k$ forms ϕ^{ν}. We want the number of forms in the system f^{ν} of the type $[g^{\lambda}]^{\mu}\cdot h^{\nu-k-\lambda}$. The forms g^{λ} map on an S_{λ}, the forms $h^{\nu-k-\lambda}$ on an $S_{\nu-k-\lambda}$ skew to S_{λ} in an $S_{\nu-k+1}$. Taking a section by an $S_{\nu+k}$, g^{λ} and $h^{\nu-k-\lambda}$ are represented by points outside the exceptional sections $S_{\lambda-1}$ and $S_{\nu-k-\lambda-1}$. Then the $\nu-k$ apolarity conditions give rise in $S_{\nu-k}$ to $\nu-k$ spreads of order $\mu+1$ which contain $S_{\nu-k-\lambda-1}$ μ times and $S_{\lambda-1}$ once. In $S_{\nu-k}$ the index numbers of $S_{\lambda-1}$ are

$$1, -\binom{\nu-k-\lambda+1}{1}, \binom{\nu-k-\lambda+2}{2}, \ldots\ldots, (-1)^{\lambda-1}\binom{\nu-k-1}{\lambda-1};$$

those of the μ-fold $S_{\nu-k-\lambda-1}$ are

$$\mu^{\lambda+1},\ -\mu^{\lambda+2}\binom{\lambda+1}{1},\ \mu^{\lambda+3}\binom{\lambda+2}{2},\ \ldots,\ (-1)^{\nu-k-\lambda-1}\mu^{\nu-k}\binom{\nu-k-1}{\nu-k-\lambda-1}.$$

Since the two manifolds have no point in common, we have

$$\begin{aligned} O = (\mu+1)^{\nu-k} &- \Big[\binom{\nu-k}{\lambda-1}(\mu+1)^{\lambda-1} - \binom{\nu-k}{\lambda-2}(\mu+1)^{\lambda-2}\binom{\nu-k-\lambda+1}{1} \\ &\qquad + \ldots + (-1)^{\lambda-1}\binom{\nu-k}{0}(\mu+1)^{0}\binom{\nu-k-1}{\lambda-1}\Big] \\ &- \Big[\binom{\nu-k}{\nu-k-\lambda-1}(\mu+1)^{\nu-k-\lambda-1}\mu^{\lambda+1} \\ &\qquad - \binom{\nu-k}{\nu-k-\lambda-2}(\mu+1)^{\nu-k-\lambda-2}\mu^{\lambda+2}\binom{\lambda+1}{1} + \ldots \\ &\qquad + (-1)^{\nu-k-\lambda-1}\binom{\nu-k}{0}(\mu+1)^{0}\mu^{\nu-k}\binom{\nu-k-1}{\nu-k-\lambda-1}\Big]. \end{aligned}$$

Applying 11° to these brackets, they become respectively

$$\left[\binom{\nu-k}{\lambda-1}\mu^{\lambda-1} + \binom{\nu-k}{\lambda-2}\mu^{\lambda-2} + \ldots + \binom{\nu-k}{0}\mu^{0}\right]$$

and

$$\left[\binom{\nu-k}{\lambda+1} + \binom{\nu-k}{\lambda+2}\mu + \ldots + \binom{\nu-k}{0}\mu^{\nu-k-\lambda-1}\right]\mu^{\lambda+1}.$$

Hence the sum of the two is $(\mu+1)^{\nu-k} - \mu^{\lambda}\binom{\nu-k}{\lambda}$ and $O = \mu^{\lambda}\binom{\nu-k}{\lambda}$.

(32) *The number of S_{k-1}'s which meet a rational curve of order ν in S_k λ times in μ coincident points, where $\lambda(\mu-1) = k$, is $\mu^{\lambda}\binom{\nu-k}{\lambda}$.*

The formula furnishes for a rational curve in S_2 the number of flexes and double tangents, for a rational curve in S_3 the number of hyperosculating and triple tangent planes, for a rational curve in S_4 the number of hyperosculating, doubly osculating, and quadruply tangent S_3's, etc.

The examples given above are drawn from a rather restricted field, since we have thus far developed explicitly the index numbers of regular manifolds only, or of combinations of them which have no common points. The extension of (32) to the case where the points of a section come together in any prescribed fashion can not yet be derived,* since, if more than two sets appear, the corresponding linear spreads have common points.

§ 2. *Relative Index Numbers.*

17. Given in S_n a manifold $M_r(\alpha)$ and a manifold $M_{r-k}(\gamma)$ upon it, $0 < k \leqq r$, then r spreads $f_1, \ldots, f_r$ of orders $\lambda_1, \ldots, \lambda_r$ will meet $M_r(\alpha)$ in general in $\alpha_0\sigma$ points. But if the spreads f_i all contain $M_{r-k}(\gamma)$, we define

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* I am indebted to Professor Morley for the formula which applies to this general case.

the relative index number, $(\alpha\gamma)_{r-k}$, *of* $M_{r-k}(\gamma)$ *as to* $M_r(\alpha)$ in terms of the orders λ_i and the earlier relative index numbers [which are similarly defined for successive sections of $M_r(\alpha)$ and $M_{r-k}(\gamma)$] by the equation

$$(33)\qquad O_{r-k}=\alpha_0\sigma_r-(\alpha\gamma)_0\sigma_{r-k}-(\alpha\gamma)_1\sigma_{r-k-1}-(\alpha\gamma)_2\sigma_{r-k-2}-\dots\dots-(\alpha\gamma)_{r-k},$$

where O_{r-k} is the number of points of $M_r(\alpha)$ outside of $M_{r-k}(\gamma)$ cut out by the spreads f_i, and the σ's refer to the orders λ. In particular, $(\alpha\gamma)_0$ is the order γ_0 of $M_{r-k}(\gamma)$. We shall now prove that

(34) *The relative index numbers of* $M_{r-k}(\gamma)$ *as to* $M_r(\alpha)$ *are independent of the spreads used to define them, and also of the dimension in which the base spread* $M_r(\alpha)$ *lies. The order of* A RELATIVE RESTRICTED SYSTEM OF EQUATIONS *is given by* (33) *in terms of the orders of the equations and the relative index numbers.*

For if λ_1 increase to λ_1+1 by using the spread $f_1\cdot L$, where L is a general S_{n-1}, then, from a section by L, O_{r-k} is increased by

$$O'_{r-k-1}=\alpha_0\sigma'_{r-1}-(\alpha\gamma)_0\sigma'_{r-k-1}-(\alpha\gamma)_1\sigma'_{r-k-2}-\dots\dots-(\alpha\gamma)_{r-k-1},$$

where the σ''s refer to $\lambda_2,\dots\dots,\lambda_n$. But this is precisely the increase on the right of (33) due to the change in λ_1; *i. e.*, $(\alpha\gamma)_{r-k}$ is unaltered and is independent of the orders. The same is true of the earlier index numbers. Again, if $M_r(\alpha)$ is supposed to lie in an S_{n+1} containing S_n, then, in S_{n+1}, spreads of the same orders λ determine the same number O_{r-k} of points of $M_r(\alpha)$ outside of $M_{r-k}(\gamma)$.

Evidently the index numbers α of $M_r(\alpha)$ in S_n, as defined in § 1, are merely the relative index numbers of $M_r(\alpha)$ as to S_n. We might define the *absolute index numbers* of M_r to be the relative index numbers of M_r as to the linear space of lowest dimension in which M_r lies, a linear space S_r being supposed to lie in an S_{r+1}. Thus the absolute index numbers of an S_r are

$$1,\ -1,\ \dots\dots,\ (-1)^k,\ \dots\dots,\qquad k=0,\ \dots\dots,\ r.$$

18. The theorems of § 1 can be generalized to apply to relative index numbers. The extension of (11) is:

(35) *If* $M_{r-k}(\gamma)$ *is the regular meet of* $M_r(\alpha)$ *with spreads of orders* $l_1,\dots\dots,l_k$, *the relative index numbers of* $M_{r-k}(\gamma)$ *as to* $M_r(\alpha)$ *are*

$$(\alpha\gamma)_j=(-1)^j\alpha_0\tau_kT_j,\qquad j=0,\ \dots\dots,\ r-k,$$

where τ *and* T *are the symmetric and complete symmetric polynomials in* $l_1,\dots\dots,l_k$.

For further spreads of orders $\lambda_1,\dots\dots,\lambda_{r-k}$ on $M_{r-k}(\gamma)$, the numbers $O_1,\dots\dots,O_{r-k}$ are all zero. First $(\alpha\gamma)_0=\gamma_0=\alpha_0\tau_k$; also $O_1=0=\alpha_0\tau_k\lambda_1-(\alpha\gamma)_0(\tau_1+\lambda_1)-(\alpha\gamma)_1$, whence $(\alpha\gamma)_1=-\alpha_0\tau_kT_1$. Assume $(\alpha\gamma)_j=(-1)^j\alpha_0\tau_kT_j$ for $j=0,1,\dots\dots,r-k-1$, and let the σ's refer to $l_1,\dots\dots,l_{r-k}$.

Then

$$\begin{aligned} O_{r-k} = 0 = \alpha_0 \tau_k \sigma_{r-k} &- (\alpha\gamma)_0 [\sigma_{r-k} + \sigma_{r-k-1}\tau_1 + \dots + \sigma_{r-2k}\tau_k] \\ &- (\alpha\gamma)_1 [\sigma_{r-k-1} + \sigma_{r-k-2}\tau_1 + \dots + \sigma_{r-2k-1}\tau_k] - \dots \\ &- (\alpha\gamma)_{r-k-1}(\sigma_1 + \tau_1) - (\alpha\gamma)_{r-k}. \end{aligned}$$

This equation determines $(\alpha\gamma)_{r-k}$. It is satisfied when also $(\alpha\gamma)_{r-k} = (-1)^{r-k}\tau_k T_{r-k}$; for then we find that

$$\begin{aligned} 0 = \alpha_0 \tau_k [\sigma_{r-k}(1-1) &+ \sigma_{r-k-1}(-\tau_1 + T_1) + \sigma_{r-k-2}(-\tau_2 + \tau_1 T_1 - T_2) + \dots \\ &+ (-\tau_{r-k} + \tau_{r-k-1} T_1 - \tau_{r-k-2} T_2 + \dots + (-1)^{r-k+1} T_{r-k})], \end{aligned}$$

which according to (10) is true.

The following theorem is the analogue of (17) and is similarly proved:

(36) *If* $(\alpha\gamma)_j$ *are the relative index numbers of* $M_{r-k}(\gamma)$ *as to* $M_r(\alpha)$, *the relative index numbers* $(\alpha\gamma)'_j$, *of* M_{r-k-1}, *the complete meet of* $M_{r-k}(\gamma)$ *and a spread of order* q, *as to* $M_r(\alpha)$, *are*

$$(\alpha\gamma)'_j = q(\alpha\gamma)_j - q^2(\alpha\gamma)_{j-1} + q^2(\alpha\gamma)_{j-2} - \dots + (-1)^j(\alpha\gamma)_0, \qquad j = 0, 1, \dots, r-k-1.$$

Let us state without proof that

(37) *The theorems* (18), (19), (20), (21), (22), (23), *when generalized as above, remain true for the relative index numbers of spreads on an* M_r.

19. The following three theorems are of different type from the foregoing:

(38) *The relative index numbers of* $M_{r-k}(\gamma)$ *as to the regular meet* $M_r(\alpha)$ *of* $n-r$ *spreads of orders* $\lambda_1, \dots, \lambda_{n-r}$ *in* S_n, *in terms of* λ *and the index numbers* γ *of* $M_{r-k}(\gamma)$ *in* S_n, *are*

$$(\alpha\gamma)_j = \gamma_j + \gamma_{j-1}\sigma_1 + \gamma_{j-2}\sigma_2 + \dots + \gamma_0\sigma_j, \quad j = 0, 1, \dots, r-k,$$

where the σ*'s refer to the* λ*'s.*

Let $M_{r-k}(\gamma)$ and its sections be on as many spreads of orders $l_1, l_2, \dots,$ in addition to those of order λ, as are necessary to determine them, and let the τ's refer to symmetric combinations of the l's. First $(\alpha\gamma)_0 = \gamma_0$; then $O_1 = \sigma_{n-r}\tau_{k+1} - \gamma_0(\sigma_1 + \tau_1) - \gamma_1 = \sigma_{n-r}\tau_{k+1} - (\alpha\gamma)_0\tau_1 - (\alpha\gamma)_1$, whence $(\alpha\gamma)_1 = \gamma_1 + \gamma_0\sigma_1$. Assuming the theorem true for $j = 0, 1, \dots, r-k-1$, we have

$$\begin{aligned} O_{r-k} &= \sigma_{n-r}\tau_r - \gamma_0(\sigma_{r-k} + \sigma_{r-k-1}\tau_1 + \dots + \tau_{r-k}) \\ &\qquad - \gamma_1(\sigma_{r-k-1} + \sigma_{r-k-2}\tau_1 + \dots + \tau_{r-k-1}) - \dots - \gamma_{r-k} \\ &= \sigma_{n-r}\tau_r - \gamma_0\tau_{r-k} - (\gamma_1 + \sigma_1\gamma_0)\tau_{r-k-1} - \dots \\ &\qquad - (\gamma_{r-k} + \gamma_{r-k-1}\sigma_1 + \dots + \gamma_0\sigma_{r-k}) \\ &= \sigma_{n-r}\tau_r - (\alpha\gamma)_0\tau_{r-k} - (\alpha\gamma)_1\tau_{r-k-1} - \dots - (\alpha\gamma)_{r-k}. \end{aligned}$$

Hence $(\alpha\gamma)_{r-k} = \gamma_{r-k} + \gamma_{r-k-1}\sigma_1 + \dots + \sigma_{r-k}$.

The generalization of this to the case where the γ's are relative index numbers is:

(39) *If* $(\alpha\gamma)_j$ *are the relative index numbers of* $M_{r-k}(\gamma)$ *as to* $M_r(\alpha)$, *the relative index numbers* $(\alpha_i\gamma)_j$ *of* $M_{r-k}(\gamma)$ *as to* M_{r-i} $(i<k)$, *the regular meet of* $M_r(\alpha)$ *and i spreads of orders* $\lambda_1, \dots, \lambda_i$, *all on* $M_{r-k}(\gamma)$, *are*

$$(\alpha_i\gamma)_j = (\alpha\gamma)_j + (\alpha\gamma)_{j-1}\sigma_1 + (\alpha\gamma)_{j-2}\sigma_2 + \dots + (\alpha\gamma)_{j-i}\sigma_i,\quad j = 0, 1, \dots, r-k,$$

where the σ*'s refer to* $\lambda_1, \dots, \lambda_i$.

The following theorem generalizes the notion of a linear section:

(40) *If* $(\alpha\gamma)_i$ *are the relative index numbers of* $M_{r-k}(\gamma)$ *as to* $M_r(\alpha)$, *and if* $M_r(\alpha)$ *and* $M_{r-k}(\gamma)$ *be cut by a spread of order* q *in* M_{r-1} *and* M_{r-k-1} *respectively, then the relative index numbers of* M_{r-k-1} *as to* M_{r-1} *are*

$$(\alpha\gamma)'_j = q\,(\alpha\gamma)_j,\quad j = 0, 1, \dots, r-k-1.$$

This is proved by finding from (36) the relative index numbers of M_{r-k-1} as to $M_r(\alpha)$ and then using (39) to get the required relative index numbers.

The theorem (14) of § 1 can be generalized as follows:

(41) *Given an* $M_r(\alpha)$ *which contains an* $M_{r-j}(\beta)$ *which contains an* $M_{r-k}(\gamma)$, $k>j$; *then*

$$\beta_0(\alpha\gamma)_i = (\alpha\beta)_0(\beta\gamma)_i + (\alpha\beta)_1(\beta\gamma)_{i-1} + \dots + (\alpha\beta)_i(\beta\gamma)_0,\quad i = 0, 1, \dots, r-k,$$

provided either $M_{r-j}(\beta)$ *is a regular meet of* $M_r(\alpha)$, *or* $M_{r-k}(\gamma)$ *is a regular meet of* $M_{r-j}(\beta)$.

For in the first case we can solve the formulæ given in (39) for $(\alpha\gamma)_\nu$ in terms of $(\alpha_i\gamma)_\nu = (\beta\gamma)_\nu$ and substitute the known relative index numbers $(\alpha\beta)_\mu$. In the second case (36) can be repeatedly applied, and the known relative index numbers $(\beta\gamma)_\nu$ can be substituted.

That the provisos in (41) are essential is shown by the following example: On $M_r(\alpha) = S_5$ let us define the $M_{r-j} = M_3(\beta)$ by the matrix $\overline{123} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = 0$, where the variables represent general linear forms in S_5. On $M_{r-j}(\beta)$ let us define $M_{r-k}(\gamma) = M_2(\gamma)$ by $x_1 = y_1 = \overline{23} = 0$. Neither $M_3(\beta)$ nor $M_2(\gamma)$ is regular in the sense of (41), and the formulæ of (41) do *not* hold. In fact the index numbers $(\alpha\beta)_i$ are the index numbers β_i in S_5. For β_0 we have in S_2 two conics, $\overline{12} = 0$, $\overline{13} = 0$, on the outside point $x_1 = y_1 = 0$, whence $\beta_0 = 3$. In S_3, S_4 and S_5, $M_3(\beta)$ is a complete spread determined by quadrics, whence $\beta_1 = -10$, $\beta_2 = 24$, $\beta_3 = -48$. Evidently $\gamma_0 = 2$. In S_4, $x_1 = y_1 = 0$ are two S_3's on $M_1(\gamma)$ which meet $M_2(\beta)$ only where $\overline{23} = 0$, *i. e.*, in no points outside $M_1(\gamma)$, whence $0 = 3\cdot1\cdot1 - (1+1)2 - (\beta\gamma)_1$, or $(\beta\gamma)_1 = -1$. In S_4, $M_1(\gamma)$ is a conic, so that its index numbers are $(\alpha\gamma)_0 = 2$, $(\alpha\gamma)_1 = -8$. Thus we have $\gamma_0 = (\alpha\gamma)_0 = 2$, $(\alpha\gamma)_1 = -8$, $\beta_0 = (\alpha\beta)_0 = 3$, $(\alpha\beta)_1 = -10$, $(\beta\gamma)_1 = -1$. The equation $\beta_0\,(\alpha\gamma)_0 = (\alpha\beta)_0\,(\beta\gamma)_0$ of course is true, but $\beta_0\,(\alpha\gamma)_1 = (\alpha\beta)_0\,(\beta\gamma)_1 + (\alpha\beta)_1\,(\beta\gamma)_0$ is not true.

On the other hand, theorem (18), which has thus far been proved only for the case where at least one manifold is regular, holds in the following example, where neither is regular. In S_5 let $M_3(\alpha)$ be defined by $\overline{123} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = 0$ and $M_3(\beta)$ by $\overline{123'} = \begin{vmatrix} z_1 & z_2 & z_3 \\ t_1 & t_2 & t_3 \end{vmatrix} = 0$, the variables representing S_4's in general position. They meet in a curve $M_1(\gamma)$ whose index numbers, according to (18), are $\gamma_0 = \alpha_0\,\beta_0 = 9$ and $\gamma_1 = \alpha_0\,\beta_1 + \alpha_1\,\beta_0 = -60$. To verify this, note that the two quadrics $\overline{12} = 0$ and $\overline{13} = 0$ meet in $M_3(\alpha)$ and the S_3, $x_1 = y_1 = 0$; also that $\overline{12'} = 0$ and $\overline{13'} = 0$ meet in $M_3(\beta)$ and the S_3, $z_1 = t_1 = 0$. The four quadrics meet in a composite curve whose index numbers are 16, $-16 \cdot 8$, This composite curve has the four parts:

(a) $\overline{123} = 0$, $\overline{123'} = 0$, with index numbers $\gamma_0 = 9$, γ_1;
(b) $\overline{123} = 0$, $z_1 = t_1 = 0$, with index numbers 3, -16;
(c) $x_1 = y_1 = 0$, $\overline{123'} = 0$, with index numbers 3, -16;
(d) $x_1 = y_1 = 0$, $z_1 = t_1 = 0$, with index numbers 1, -4.

But the pairs (a), (b) and (a), (c) each have six common points; the pairs (b), (d) and (c), (d) each have two common points; while the pairs (a), (d) and (b), (c) have no common points. Hence the index numbers of the four, according to (24), are 16, $(\gamma_1 - 16 - 16 - 4 - 2 \cdot 16)$. Comparing these with 16, $-16 \cdot 8$, we find that $\gamma_1 = -60$, which verifies the theorem.

These examples emphasize the fact that considerable caution must be used in accepting as general theorems which have been proved only for the cases where some or all of the manifolds concerned are regular.

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